Transformations Between Two Images

- Translation
- Rotation
- Rigid
- Similarity (scaled rotation)
- Affine
- Projective
- Pseudo Perspective
- Bi-linear

Fundamental Matrix

Lecture-12

Applications

- Stereo
- Structure from Motion
- View Invariant Action Recognition
- ..

Stereo Pairs and Depth Maps (from Szeliski's book)

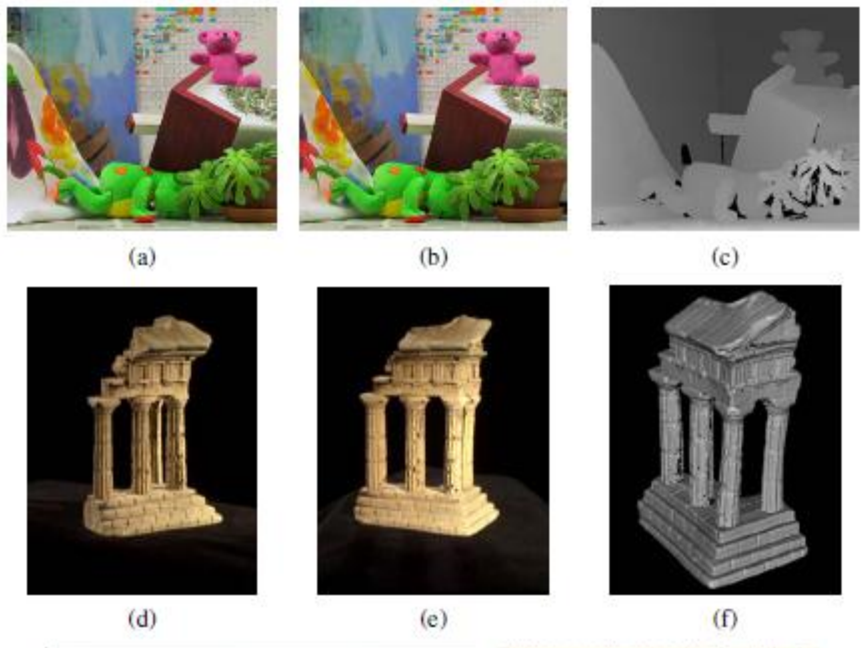
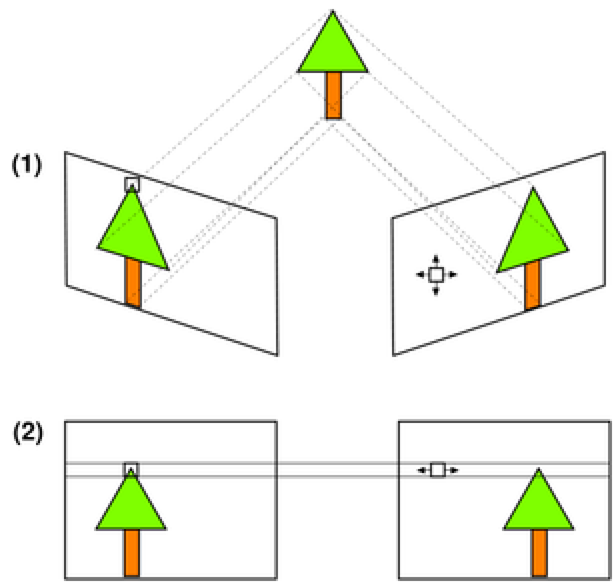
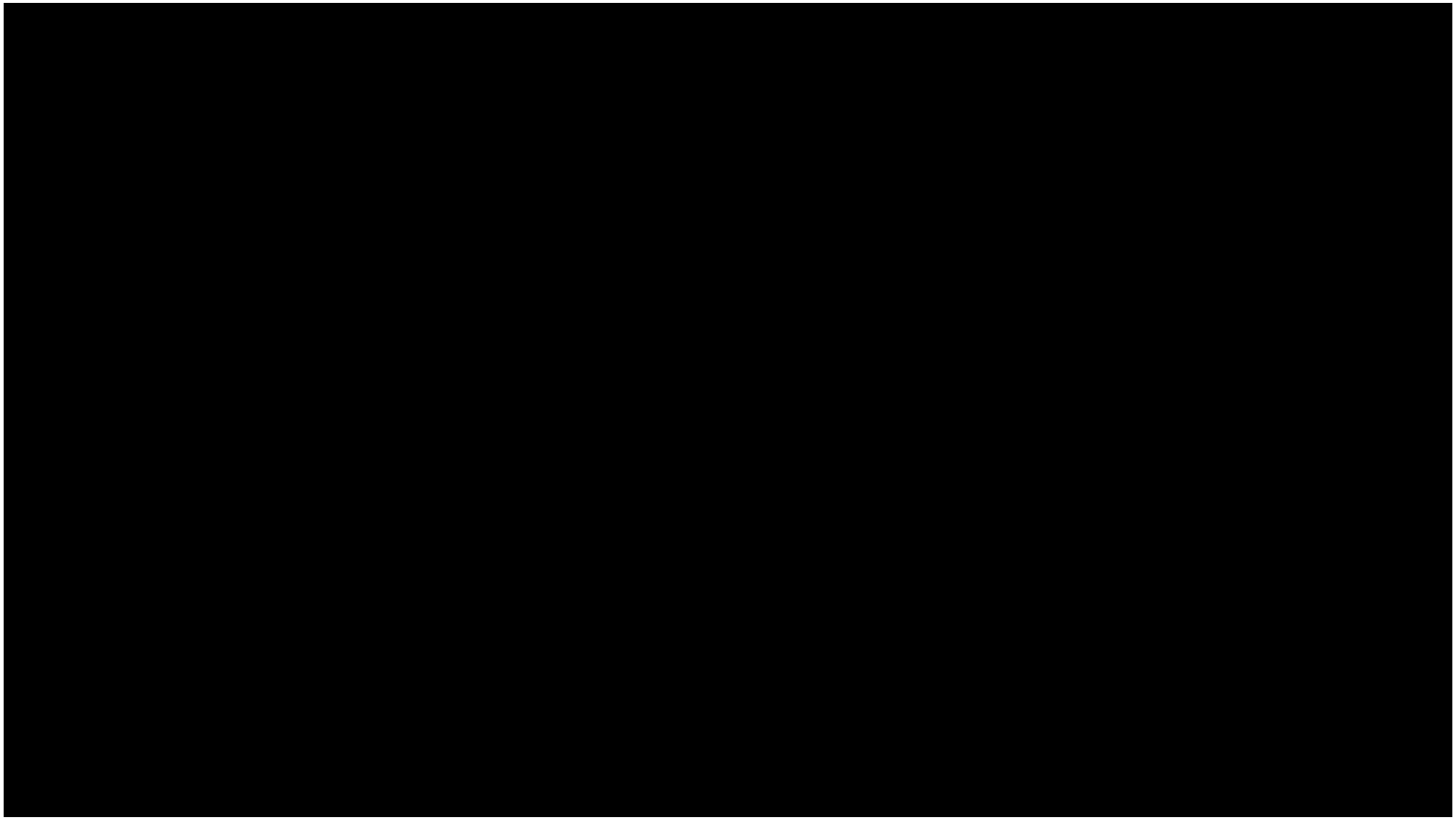


Image Rectification For Stereo



Photosynth: Structure From Motion



Fundamental Matrix

- Longuet Higgins (1981)
- Hartley (1992)
- Faugeras (1992)
- Zhang (1995)

Fundamental Matrix Song

http://www.youtube.com/watch?v=DgGV3l82NTk

Preliminaries

- Linear Independence
- Rank of a Matrix
- Matrix Norm
- Singular Value Decomposition
- Vector Cross product to Matrix Multiplication
- RANSAC

Linearly Independence

A finite subset of n vectors, $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$, from the vector space V, is ***linearly dependent*** if and only if there exists a set of n scalars, a_1, a_2, ..., a_n, not all zero, such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}.$$

Rank of a Matrix

- The **column rank** of a matrix A is the maximum number of linearly independent column vectors of A.
- The **row rank** of a matrix A is the maximum number of linearly independent row vectors of A.
- The column rank of A is the dimension of the column space of A
- The row rank of A is the dimension of the row space of A.

Example (Row Echelon)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} R_2 \to 2r_1 + r_2$$

Rank is 2

Singular Value Decomposition (SVD)

Theorem: Any *m* by *n* matrix A, for which $m \geq n$,can be written as

$$A = O_1 \Sigma O_2$$

mxn mxn nxn nxn

Σ is diagonal

O_1 ,O_2 are orthogonal

$$O_1^T O_1 = O_2^T \, O_2 = I$$

Matrix Norm

L1 matrix norm is maximum of absolute column sum.

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|,$$

Vector Cross Product to Matrix-vector multiplication

$$A \times B = \begin{bmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix} = \begin{bmatrix} -A_z B_y + A_y B_z & A_z B_x - A_x B_z & -B_x A_y + B_y A_x \end{bmatrix}$$

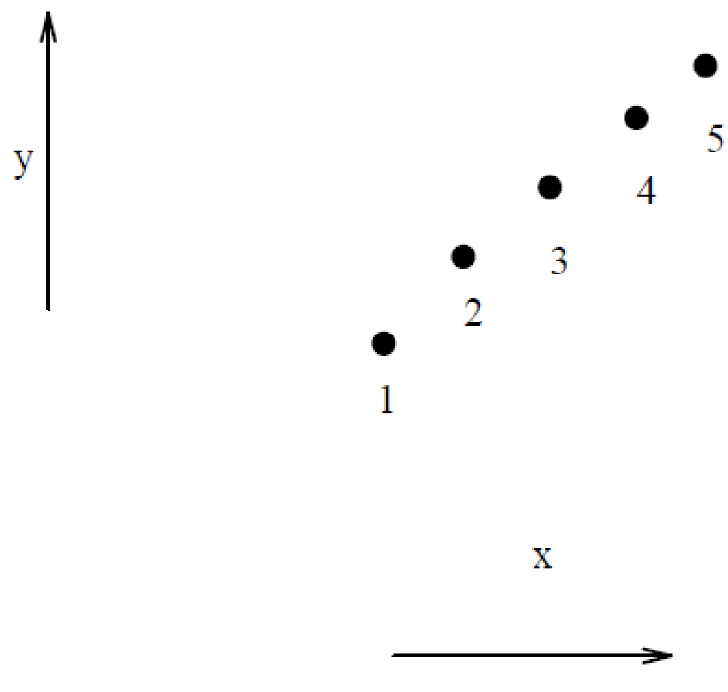
$$A \times B = S.B = \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} -A_z B_y + A_y B_z & A_z B_x - A_x B_z & -B_x A_y + B_y A_x \end{bmatrix}$$

RANSAC: Random Sampling and Consensus

RANSAC Song

http://www.youtube.com/watch?v=1YNjMxxXO-E&feature=relmfu

How to Fit A Line?



$$y = mx + c$$

How to Fit A Line?

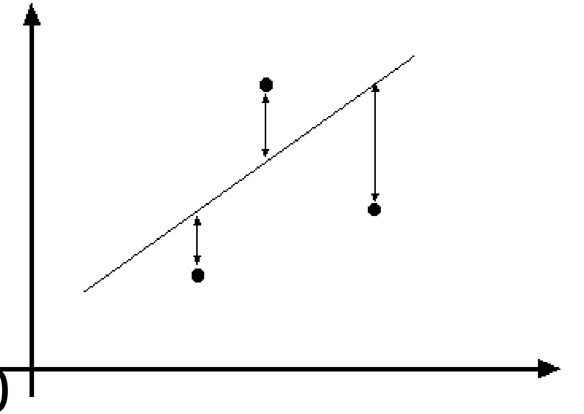
- Least squares Fit (over constraint)
- RANSAC (constraint)
- Hough Transform (under constraint)

Least Squares Fit

- Standard linear solution to estimating unknowns.
  - If we know which points belong to which line
  - Or if there is only one line

$$y = mx + c = f(x, m, c)$$

$$\text{Minimize } E = \sum_i \left[y_i - f(x_i, m, c)\right]^2$$



Take derivative wrt *m* and *c* set to 0

Alper Yilmaz, Mubarak Shah, Fall 2011 UCF

Line Fitting

$$y = mx + c$$

$$y_1 = mx_1 + c$$
$$y_2 = mx_2 + c$$
$$\vdots$$
$$y_n = mx_n + c$$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & 1 \\ x_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ c \end{bmatrix}}_{D} \quad \Rightarrow B = AD$$

$$A^T B = A^T A D$$
$$\left(A^T A\right)^{-1} A^T B = \left(A^T A\right)^{-1} \left(A^T A\right) D$$
$$D = \left(A^T A\right)^{-1} A^T B$$

RANSAC: Random Sampling and Consensus

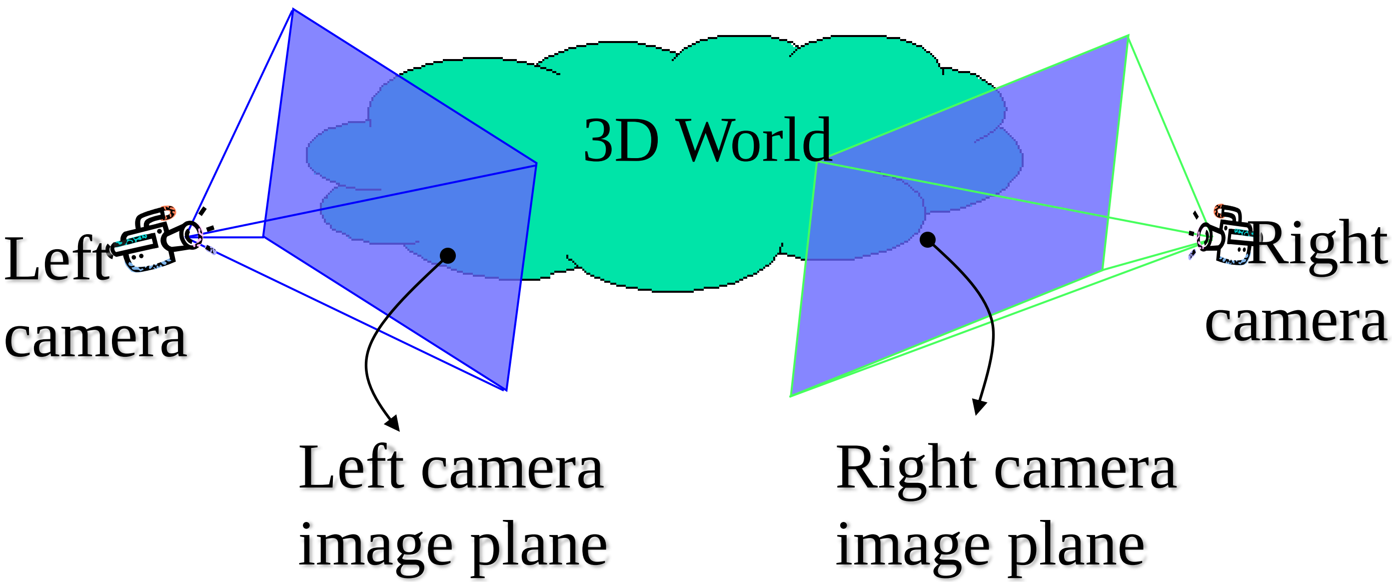
1. Randomly select two points to fit a line

2. Find the error between the estimated solution an all other points. If the error is less than tolerance, then quit, else go to step (1).

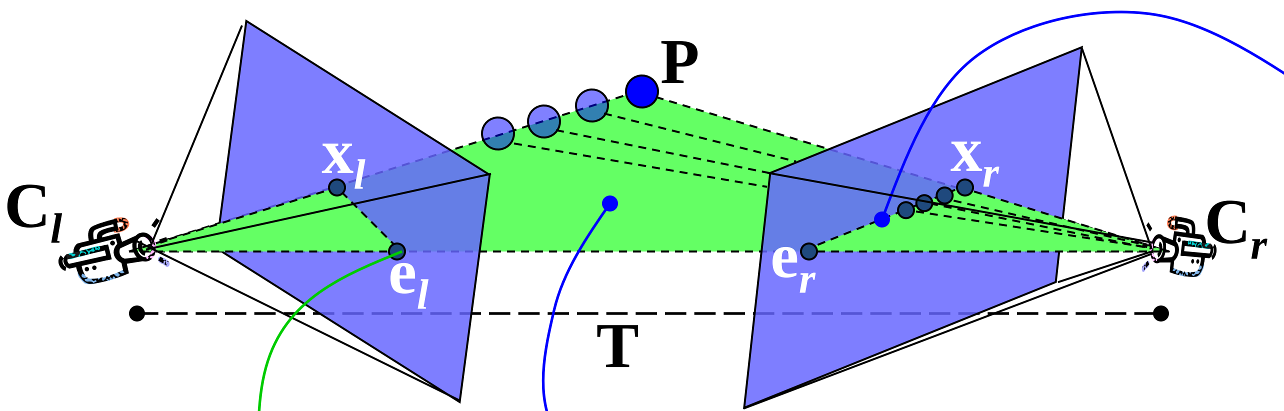
Derivation of Fundamental Matrix

Epipolar Geometry

- Defined for two static cameras



Epipolar Geometry

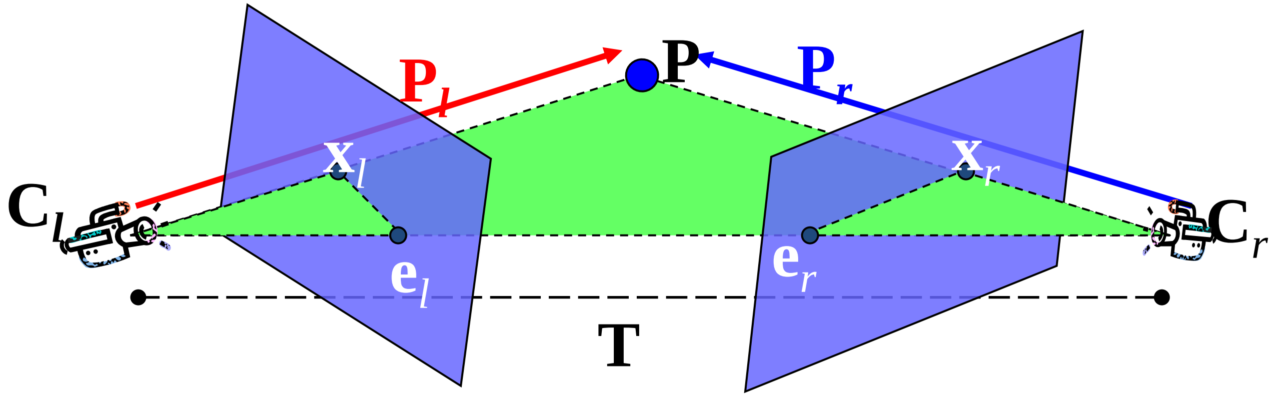


Epipolar line: set of world points that project to the same point in left image, when projected to right image forms a line.

Epipolar plane: plane defined by the camera centers and world point.

Epipole: intersection of image plane with line connecting camera centers. Image of a left camera center in the right, and vice versa.

Essential Matrix



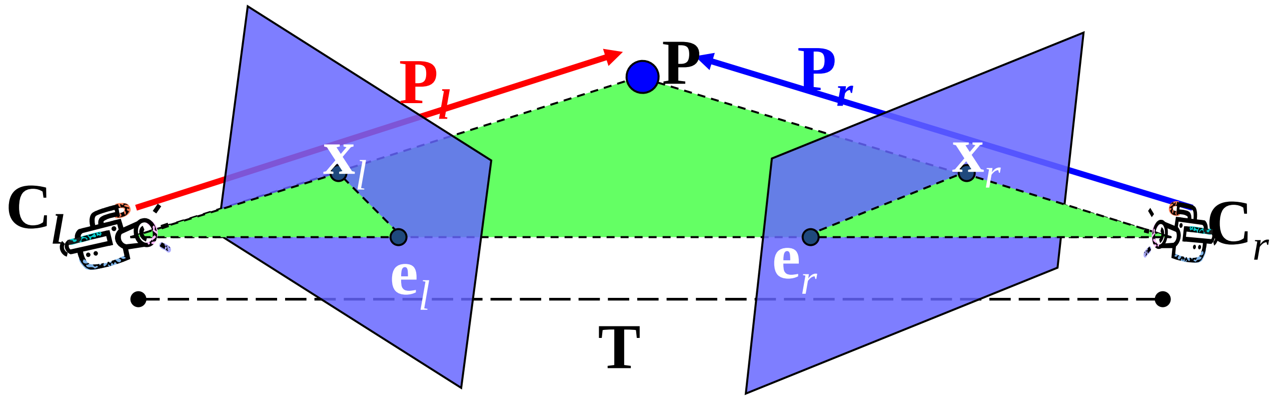
Coplanarity constraint between vectors ($\mathbf{P}_l$-**T**), **T**, $\mathbf{P}_l$.

$$\left.\begin{aligned}(\mathbf{P}_l - \mathbf{T})^{\mathrm{T}}\mathbf{T}\times\mathbf{P}_l = 0 \\ \mathbf{P}_r = \mathbf{R}(\mathbf{P}_l - \mathbf{T})\end{aligned}\right\} \qquad \mathbf{P}_r^{\ \mathrm{T}}\mathbf{R}\mathbf{T}\times\mathbf{P}_l = 0$$

$$\mathbf{R}^T\mathbf{P}_r = (\mathbf{P}_l - \mathbf{T})$$

$$\mathbf{P}_{\mathbf{r}}^{\ T}\mathbf{R} = (\mathbf{P}_l - \mathbf{T})^T$$

Essential Matrix



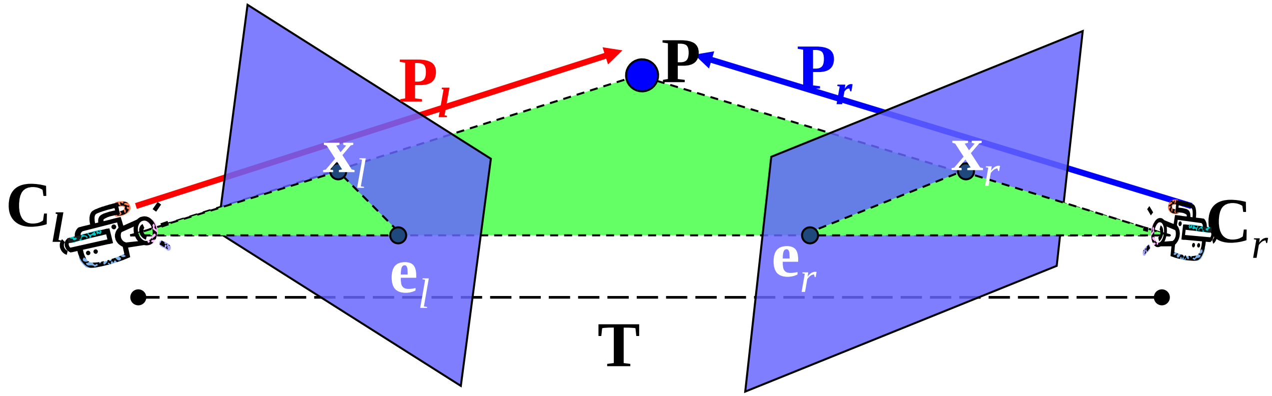
$$\left.\mathbf{P}_r^{\mathrm{T}}\mathbf{R}\mathbf{T}\times\mathbf{P}_l = 0\right\} \quad \mathbf{P}_r^{\mathrm{T}}\mathbf{R}\,\boxed{\mathbf{S}}\,\mathbf{P}_l = 0 \quad \Rightarrow \quad \boxed{\mathbf{P}_r^{\mathrm{T}} E \mathbf{P}_l = 0}$$

essential matrix

$$\boxed{\begin{pmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{pmatrix}}$$

$$E = \mathbf{R}\ \mathbf{S}$$

Fundamental Matrix



Apply Camera model

$$M_l^{-1}\mathbf{x}_l = \mathbf{P}_l$$

$$M_r^{-1}\mathbf{x}_r = \mathbf{P}_r$$

$$\mathbf{x}_r^T M_r^{-T} = \mathbf{P}_r^T$$

$$\left.\begin{array}{l} \mathbf{x}_l = M_l \mathbf{P}_l \\ \mathbf{x}_r = M_r \mathbf{P}_r \\ \mathbf{P}_r^{\mathrm{T}} E \mathbf{P}_l = 0 \end{array}\right\}$$

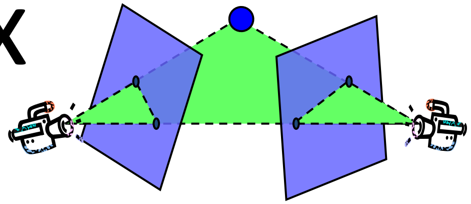
$$\mathbf{x}_r^{\mathrm{T}} M_r^{-\mathrm{T}} E M_l^{-1} \mathbf{x}_l = 0$$

$$\mathbf{x}_r^{\mathrm{T}} \left(M_r^{-\mathrm{T}} E M_l^{-1}\right) \mathbf{x}_l = 0$$

$$\boxed{\mathbf{x}_r^{\mathrm{T}} F \mathbf{x}_l = 0}$$

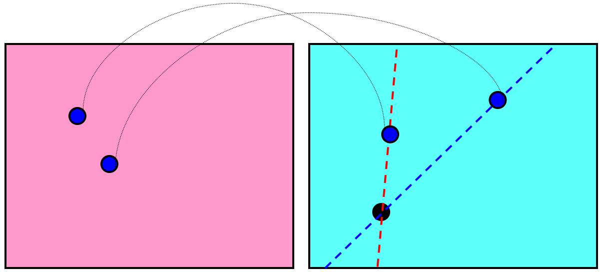
fundamental matrix

Fundamental Matrix

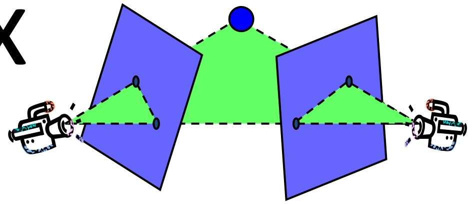


$$\mathbf{x}'^{\mathrm{T}} F\mathbf{x} = \mathbf{x}'^{\mathrm{T}} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & m \end{pmatrix} \mathbf{x} = 0$$

- Given a point in left camera $\mathbf{x}$, epipolar line in right camera is: $\mathbf{u}_r = F\mathbf{x}$



Fundamental Matrix



$$\boxed{\mathbf{x}_l^{\mathrm{T}} F \mathbf{x}_r = \mathbf{x}_l^{\mathrm{T}} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & m \end{pmatrix} \mathbf{x}_r = 0}$$

- 3x3 matrix with 9 components
- Rank 2 matrix (due to S)
- 7 degrees of freedom
- Given a point in left camera $\mathbf{x}$, epipolar line in right camera is: $\mathbf{u}_r = F\mathbf{x}$

Fundamental Matrix

- Longuet Higgins (1981)
- Hartley (1992)
- Faugeras (1992)
- Zhang (1995)

Fundamental Matrix

- Fundamental matrix captures the relationship between the corresponding points in two views.

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}^T \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} x_i' \\ y_i' \\ 1 \end{bmatrix} = 0,$$

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}^T \begin{bmatrix} f_{11}x' + f_{12}y_i' + f_{13} \\ f_{21}x' + f_{22}y_i' + f_{23} \\ f_{31}x' + f_{32}y_i' + f_{33} \end{bmatrix} = 0,$$

Fundamental Matrix

$$x_i(f_{11}x' + f_{12}y'_i + f_{13}) + y_i(f_{21}x' + f_{22}y'_i + f_{23}) + (f_{31}x' + f_{32}y'_i + f_{33}) = 0$$

$$x_i x' f_{11} + x_i y'_i f_{12} + x_i f_{13} + y_i x' f_{21} + x' y'_i f_{22} + y'_i f_{23} + x' f_{31} + y'_i f_{32} + f_{33} = 0$$

One equation for one point correspondence

$$Mf = \begin{bmatrix} x'_1 x_1 & x'_1 y_1 & x'_1 & y'_1 x_1 & y'_1 y_1 & y'_1 & x_1 & y_1 & 1 \\ x'_2 x_2 & x'_2 y_2 & x'_2 & y'_2 x_2 & y'_2 y_2 & y'_2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_n x_n & x'_n y_n & x'_1 & y'_n x_n & y'_n y_n & y'_n & x_n & y_n & 1 \end{bmatrix} f = 0$$

M is 9 by n matrix $\quad f = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{21} & f_{22} & f_{23} & f_{31} & f_{32} & f_{33} \end{bmatrix}$

To solve the equation, the rank(M) must be 8.

Computation of Fundamental Matrix

Normalized 8-point algorithm (Hartley)

Objective:

Compute fundamental matrix F such that $\mathbf{x}_i' F \mathbf{x}_i = 0$

Algorithm

Normalize the image $\hat{\mathbf{x}}_i = T\mathbf{x}_i$ $\hat{\mathbf{x}}_i' = T'\mathbf{x}_i'$

$$T = \begin{bmatrix} a_x & 0 & d_x \\ 0 & a_y & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

Find centroid of points in each image, determine the range, and normalize all points between 0 and 1

Linear solution

determining the eigen vector corresponding to the smallest eigen value of A,

$$Af = \begin{bmatrix} \hat{x}_1'\hat{x}_1 & \hat{x}_1'\hat{y}_1 & \hat{x}_1' & \hat{y}_1'\hat{x}_1 & \hat{y}_1'\hat{y}_1 & \hat{y}_1' & \hat{x}_1 & \hat{y}_1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hat{x}_8'\hat{x}_8 & \hat{x}_8'\hat{y}_8 & \hat{x}_8' & \hat{y}_8'\hat{x}_8 & \hat{y}_8'\hat{y}_8 & \hat{y}_8' & \hat{x}_8' & \hat{y}_8 & 1 \end{bmatrix} f = 0$$

Normalized 8-point algorithm (Hartley)

Construct

$$\hat{F} = \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix}$$

Normalize

$$\hat{F} = \hat{F} / \| \hat{F} \|$$

L1 matrix norm is maximum of absolute column sum.

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|,$$

Constraint enforcement SVD decomposition

$$\hat{F} = U \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} V' \qquad (\sigma_1 \geq \sigma_2 \geq \sigma_3)$$

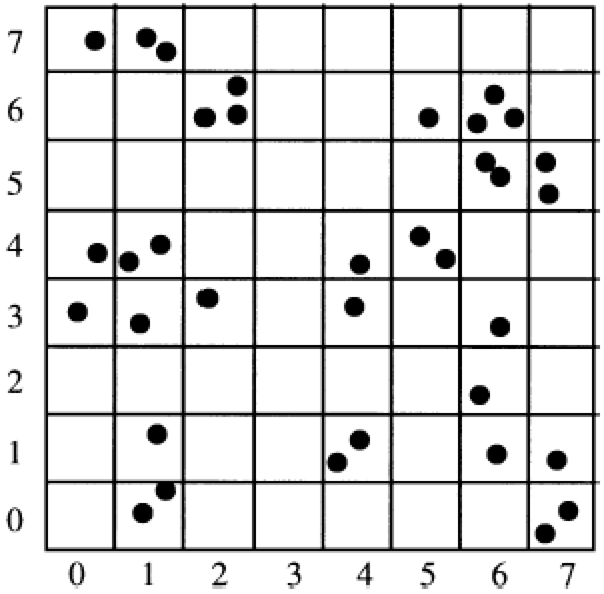
Rank enforcement

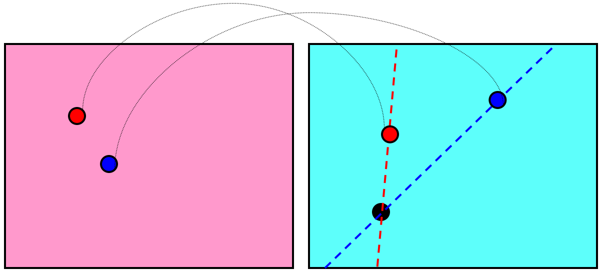
$$\widetilde{F} = U \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V' \qquad (\sigma_3 = 0)$$

De-normalization: $F = T'^{T} \widetilde{F} T$

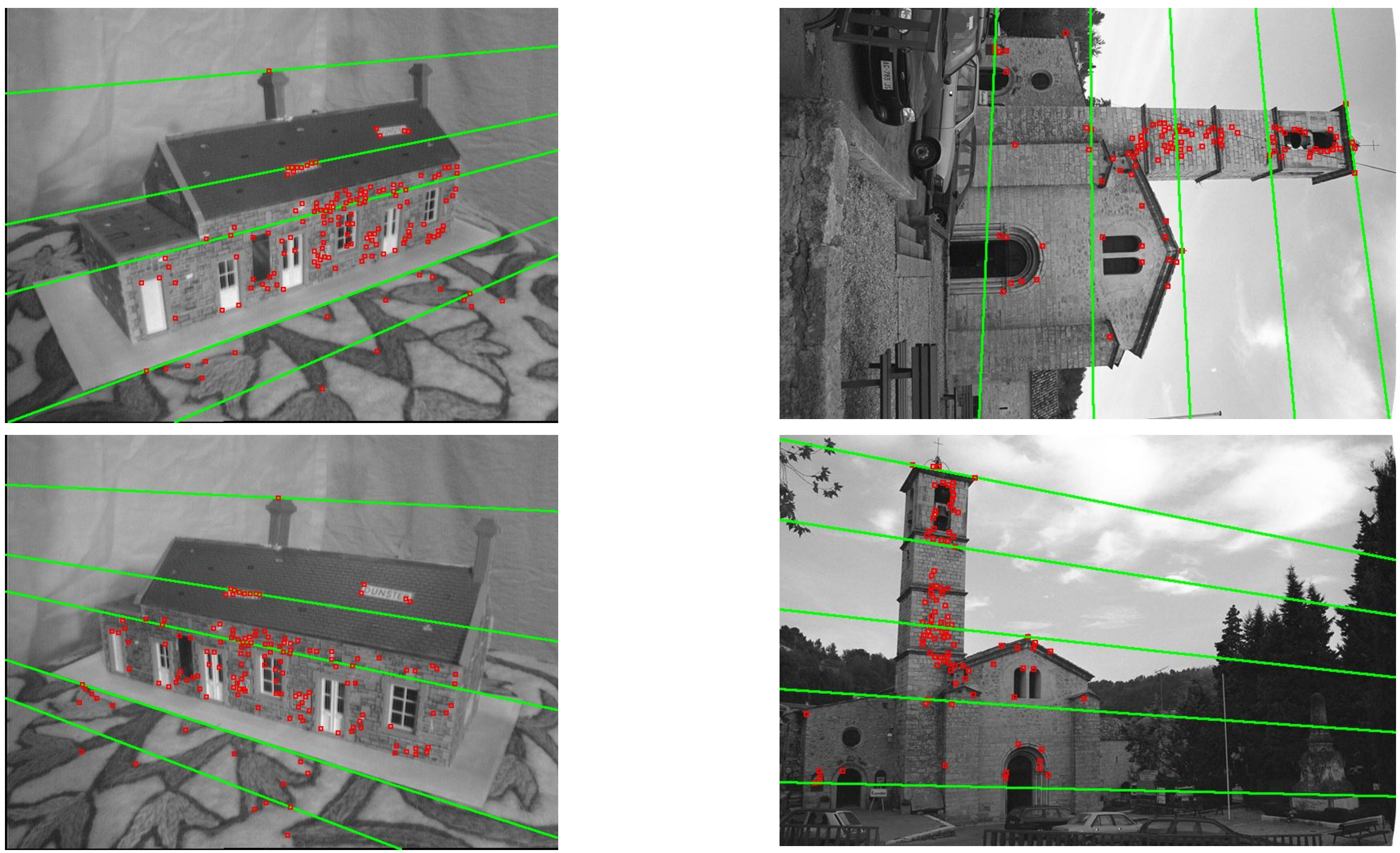
Robust Fundamental Matrix Estimation (by Zhang)

- Uniformly divide the image into 8×8 grid.
- Randomly select 8 grid cells and pick one pair of corresponding points from each grid cell, then use Hartley's 8-point algorithm to compute Fundamental Matrix F_i.
- For each F_i, compute the median of the squared residuals R_i.
  - $R_i = \text{median}_k[d(p_{1k}, F_i p_{2k}) + d(p_{2k}, F'_i p_{1k})]$
- Select the best F_i according to R_i.
- Determine outliers if $R_k > Th$.
- Using the remaining points compute the fundamental Matrix F by weighted least square method.





Epi-polar Lines



Epi-polar lines